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TRANSFORMATION OPERATORS FOR THE PERTURBED HILL EQUATION WITH COMPLEX COEFFICIENTS

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Abstract. In this paper, we consider the perturbed Hill equation on the entire axis. Using a transformation operator that preserves asymptotics at infinity, triangular representations of special solutions of this equation are found. Estimates for the representation kernels are obtained.

Keywords: perturbed equation, Hill equation, Floquet solutions, transformation operator, Bessel function, Riemann function.

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1 Introduction and main result

It is well known that the concept of a transformation operator is a generalization of the concept of similar matrices (see Levitan, (1987)). Let E be a topological linear space and A, B be two linear, but not necessarily continuous, operators from E to E_2, where E_1 and E_2 are closed subspaces in E .

Definition 1. *A continuous linear operator X defined on the whole space E, and acting E_1 to E_2 is called a transformation operator for a pair of operators A and B, if it satisfies the following two conditions:*

1) X has a continuous inverse X^{-1} on the whole of E;

2) There is an operator equation

$$AX = XB.$$

In the investigation of direct and inverse scattering problems transformation operators play an important role. They first showed up in the context of generalized shift operators in the work of Delsarte (1938). They have also been investigated by Levitan (1956) and were constructed for arbitrary Sturm–Liouville equations by Povzner (1948). Afterwards transformation operators

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have been applied for the first time when considering inverse spectral problems, for example by Marchenko (1973), who noticed that the spectral function of a Sturm–Liouville operator determines the operator uniquely in Marchenko(1986) . Soon after that Gel'fand and Levitan (1962) found a method of recovering Sturm–Liouville equations from its spectral functions, using the transformation operator techniques. Another important step was the introduction of transformation operators, which preserve the asymptotic behavior of solutions at infinity by Levin (1956). Since that these transformation operators are the main tool for solving different kinds of scattering problems, mainly in the case of constant backgrounds (see Levitan (1987), Marchenko (1986).

Consider the perturbed Hill equation

$$-y'' + p\left(x\right)y + q\left(x\right)y = \lambda y, -\infty < x < +\infty, \tag{1}$$

where $p\left(x\right)$is a continuous periodic function (real or complex valued) and complex -valued function $q\left(x\right)$ satisfies a first moment condition, i.e.

$$\int_{-\infty}^{+\infty} \left(1 + |x|\right) \left|q\left(x\right)\right| dx < \infty. \tag{2}$$

In Boutet de Monvell et al.(2008), Firsova (1987), Khanmamedov et al.(2022), a transformation operator with a condition at infinity for equation (1) was constructed in the case when $p\left(x\right)$ is a real function. These works essentially use the eigenfunction expansion of the corresponding self-adjoint unperturbed operator, which is based on the reality of the function $p\left(x\right)$. Therefore, the methods used in Boutet de Monvell et al.(2008), Firsova (1987), Khanmamedov et al.(2022) are not applicable in the case of a complex potential $p\left(x\right)$. In this direction, we note the work Khanmamedov et al.(2023) in which a transformation operator was constructed for equation (1), when $p\left(x\right) = e^{ix}$. Spectral analysis of the Hill equation with the coefficient $p\left(x\right) = \sum\limits_{n=1}^{\infty} p_n e^{inx}$ was studied in detail in Gasymov (1980), Pastur et al.(1991).

Let us consider the equation

$$-y'' + i\ \left(\sin x\right)y + q\left(x\right)y = \lambda y, -\infty < x < +\infty, \tag{3}$$

where the complex-valued function $q(x)$ satisfies the condition (2). In this paper, the transformation operators with conditions at infinity were constructed for equation (1). Unlike work Khanmamedov et al.(2023), in our case the periodic potential has an infinite number of real gaps. However, there are significant qualitative differences in the band structure of these potentials compared to ordinary real periodic potentials. In fact, although the potential $p\left(x\right) = i \sin x$ has an infinite number of gaps; there are periodic wave functions at the edges of the zone, but no anti-periodic wave functions (see see Bender et al.(1999). In this regard, the results obtained can be useful in studying direct and inverse scattering problems for equation (1).

Note that $q(x) = 0$ the equation (1.3) is the Mathieu equation, which describes the vibration of elliptical drumheads. Mathieu's equation also has various applications in many fields of physical sciences, such as optics, quantum mechanics and general relativity (see Derek (2020), Kovacic et al.(2018)).

Let $\varphi\left(x, \lambda\right)$ and $\theta\left(x, \lambda\right)$ be the solutions of equation

$$-y'' + i\left(\sin x\right)y = \lambda y, -\infty < x < +\infty, \tag{4}$$

such that $\varphi\left(0, \lambda\right) = 0$, $\varphi'\left(0, \lambda\right) = 1$,$\theta\left(0, \lambda\right) = 1$, $\theta'\left(0, \lambda\right) = 0$. Then for any given x $\varphi\left(x, \lambda\right)$, $\varphi'\left(x, \lambda\right)$,$\theta\left(x, \lambda\right)$, $\theta'\left(x, \lambda\right)$ are entire functions of λ. We set $F\left(\lambda\right) = \frac{\varphi'(\pi,\lambda)+\theta(\pi,\lambda)}{2}$. Let (see, for example, Magnus (1966)) $e^{\pm ik} = F\left(\lambda\right) \pm i\sqrt{1 - F^2\left(\lambda\right)}$ and suppose that the functions $\Psi_{\pm}\left(x, \lambda\right) = e^{\pm ik(2\pi)^{-1}x}\chi_{\pm}\left(x, \lambda\right)$, where $\chi_{\pm}\left(x + 2\pi, \lambda\right) = \chi_{\pm}\left(x, \lambda\right)$, are the Floquet solutions of the unperturbed equation.

We shall use the following notation

$$\sigma^{\pm}(x) = \pm \int_x^{\pm\infty} |q(t)|\, dt,\ \sigma_1^{\pm}(x) = \pm \int_x^{\pm\infty} \sigma^{\pm}(t)\, dt.$$

The main result of the present paper is as follows.

Theorem 1. *If the potential $q(x)$ satisfies condition (2), then, for all values of λ, Eq. (3) has solutions $f_{\pm}(x,\lambda)$ representable as*

$$f_{\pm}(x,\lambda) = \Psi_{\pm}(x,\lambda) \pm \int_x^{\pm\infty} K^{\pm}(x,t)\ \Psi_{\pm}(t,\lambda)\, dt, \tag{5}$$

where the kernels $K^{\pm}(x,t)$ are continuous functions and satisfy the following conditions:

$$\left|K^{\pm}(x,t)\right| \le C\sigma_0^{\pm}\left(\frac{x+t}{2}\right) e^{C\sigma_1^{\pm}(x)}, \tag{6}$$

$$K^{\pm}(x,x) = \pm\frac{1}{2}\int_x^{\pm\infty} q\ (t)\ dt, \tag{7}$$

here and below, C denotes, generally speaking, different constants.

2 Proof of the theorem

Without loss of generality, we consider the case "+" and assume that $x > 0$.

In proving the theorem, the following lemmas will be required.

Lemma 1. *Let*

$$z = \sqrt{2i(\sin\xi - \sin\xi_0)\ (\cos\eta_0 - \cos\eta)},\ \xi_0 < \xi < \infty,\ 0 < \eta < \eta_0, \tag{8}$$

$$R(\xi,\eta,\xi_0,\eta_0) = J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n}, \tag{9}$$

then the function $R(\xi,\eta,\xi_0,\eta_0)$ satisfies the equation

$$\frac{\partial^2 R}{\partial\xi\,\partial\eta} - 2i\cos\xi\sin\eta\, R = 0 \tag{10}$$

and

$$R(\xi_0,\eta;\xi_0,\eta_0) = R(\xi,\eta_0;\xi_0,\eta_0) = 1. \tag{11}$$

In other words, $R(\xi,\eta,\xi_0,\eta_0)$ is the Riemann function of the equation (10) and has the symmetric property

$$R(\xi,\eta;\xi_0,\eta_0) = R(\xi_0,\eta_0;\xi,\eta). \tag{12}$$

Proof. Differentiating (9) and taking into account (8), it is directly verified that

$$\frac{\partial^2 R}{\partial\xi\,\partial\eta} - 2i\cos\xi\sin\eta\, R = 2i\cos\xi\sin\eta\left[J_0''(z) + \frac{1}{z}J_0'(z) + J_0(z)\right] = 0.$$

From (8), (9), we have (11), (12).

Since for real x the trigonometric functions $\sin x$ and $\cos x$ vary from -1 to 1, we have $|J_0(z)| \le 1$. Further, using the well-known Abramowitz et al.(1964) relations $J_{n+1}(z) + J_{n-1}(z) = \frac{2n}{z}J_n(z)$, $J_n'(z) = -J_{n+1}(z) + \frac{nJ_n(z)}{z}$, where $J_n(z)$ is the Bessel function of the first kind, we obtain that the Riemann function (see Courant (1962)) $R(\xi,\eta,\,\xi_0,\eta_0)$ and its partial derivatives are uniformly bounded up to the second order:

$$|R(\xi_0,\eta_0,\xi,\eta)| + \left|\frac{\partial R(\xi_0,\eta_0,\xi,\eta)}{\partial \xi_0}\right| + \left|\frac{\partial R(\xi_0,\eta_0,\xi,\eta)}{\partial \eta_0}\right| +$$

$$\left|\frac{\partial^2 R(\xi_0,\eta_0,\xi,\eta)}{\partial \xi_0^2}\right| + \left|\frac{\partial^2 R(\xi_0,\eta_0,\xi,\eta)}{\partial \eta_0^2}\right| + \left|\frac{\partial^2 R(\xi_0,\eta_0,\xi,\eta)}{\partial \xi_0 \partial \eta_0}\right| \le C,\, C = const. \tag{13}$$

□

Lemma 2. *If $q(x)$ satisfies condition(2), then the integral equation*

$$U(\xi_0,\eta_0) = \frac{1}{2}\int_{\xi_0}^{\infty} R(\xi,0,\xi_0,\eta_0)\,q(\xi)\,d\xi - \int_{\xi_0}^{\infty} d\xi \int_0^{\eta_0} U(\xi,\eta)\,R(\xi,\eta,\xi_0,\eta_0)\,q(\xi-\eta)\,d\eta. \tag{14}$$

has one and only one solution $U(\xi_0,\eta_0)$. Furthermore,

$$|U(\xi_0,\eta_0)| \le C\sigma_0^+(\xi_0)\,e^{C\sigma_1^+(\xi_0-\eta_0)},\, C = const \tag{15}$$

Proof. Using the method of successive approximation, let

$$U_0(\xi_0,\eta_0) = \frac{1}{2}\int_{\xi_0}^{\infty} R(\xi,0;\xi_0,\eta_0)q(\xi)d\xi, \tag{16}$$

$$U_n(\xi_0,\eta_0) = -\int_{\xi_0}^{\infty} d\xi \int_0^{\eta_0} U_{n-1}(\xi,\eta)q(\xi-\eta)R(\xi,\eta;\xi_0,\eta_0)d\eta. \tag{17}$$

Since $\xi > \xi_0$, $\eta < \eta_0$, from relations (13), (16), we obtain

$$|U_0(\xi_0,\eta_0)| \le \frac{1}{2}\int_{\xi_0}^{\infty} |\,R(\xi,0;\xi_0,\eta_0)\,|\;|q(\xi)|\,d\xi \le C\int_{\xi_0}^{\infty} |q(\xi)|\,d\xi.$$

Hence,

$$|U_0(\xi_0,\eta_0)| \le C\sigma_0^+(\xi_0).$$

Further, using (5), we find that

$$|U_1(\xi_0,\eta_0)| \le \int_{\xi_0}^{\infty} d\xi \int_0^{\eta_0} |U_0(\xi,\eta)|\cdot|q(\xi-\eta)|\cdot R(\xi,\eta;\,\xi_0,\eta_0)\,d\eta \le$$

$$\le C^2\int_{\xi_0}^{\infty} d\xi \int_0^{\eta_0} \sigma_0^+(\xi)\;|q(\xi-\eta)|\,d\eta \le C^2\int_{\xi_0}^{\infty} \sigma_0^+(\xi)\,d\xi \int_0^{\eta_0} |q(\xi-\eta)|\,d\eta \le$$

$$\le C^2\sigma_0^+(\xi_0)\int_{\xi_0}^{\infty} d\xi \int_0^{\eta_0} |q(\xi-\eta)|\,d\eta = C^2\sigma_0^+(\xi_0)\int_{\xi_0}^{\infty} d\xi \int_{\xi-\eta_0}^{\xi} |q(\alpha)|\,d\alpha \le$$

$$\le C^2\sigma_0^+(\xi_0)\int_{\xi_0}^{\infty} d\xi \int_{\xi-\eta_0}^{\infty} |q(\alpha)|\,d\alpha \le C^2\sigma_0^+(\xi_0)\int_{\xi_0}^{\infty} \sigma_0^+(\xi-\eta_0)\,d\xi = C\sigma_0^+(\xi_0)\,C\sigma_1^+(\xi_0-\eta_0).$$

Let now

$$|U_{n-1}(\xi_0,\eta_0)| \le C\sigma_0^+(\xi_0)\,\frac{\left(C\sigma_1^+(\xi_0-\eta_0)\right)^{n-1}}{(n-1)!}.$$

In this case, we will have

$$|U_n(\xi_0,\eta_0)| \le \int_{\xi_0}^{\infty} d\xi \int_0^{\eta_0} |q(\xi-\eta)R(\xi,\eta,\,\xi_0,\eta_0)U_{n-1}(\xi,\eta)|\,d\eta \;\le$$

$$C^2\,\sigma_0^+(\xi_0)\int_{\xi_0}^{\infty}\frac{\left(C\sigma_1^+(\xi-\eta_0)\right)^{n-1}}{(n-1)!}\int_{\xi-\eta_0}^{\xi}|q(\alpha)|\,d\alpha d\xi\le$$

$$\le C^2\,\sigma_0^+(\xi_0)\int_{\xi_0}^{\infty}\frac{\left(C\sigma_1^+(\xi-\eta_0)\right)^{n-1}}{(n-1)!}\int_{\xi-\eta_0}^{\infty}|q(\alpha)|\,d\alpha d\xi\le$$

$$=-C^2\,\sigma_0^+(\xi_0)\int_{\xi_0}^{\infty}\frac{\left(C\sigma_1^+(\xi-\eta_0)\right)^{n-1}}{(n-1)!}d\sigma_1(\xi-\eta_0)=C\sigma_0^+(\xi_0)\,\frac{\left(C\sigma_1^+(\xi_0-\eta_0)\right)^{n}}{n!}.$$

Hence the series $U(\xi_0,\eta_0)=\sum\limits_{n=0}^{\infty}U_n(\xi_0,\eta_0)$ is absolutely and uniformly convergent, so $U(\xi_0,\eta_0)$ is the solution of the integral equation (14), and $U(\xi_0,\eta_0)$ satisfies the inequality (15).

□

Lemma 3. *Suppose $q(x)$ is a continuously differentiable function and satisfies the condition*

$$\int_{-\infty}^{+\infty}\left[x^2\,|q(x)|+\left|q'(x)\right|\right]dx<\infty, \tag{18}$$

then the solution $U(\xi_0,\eta_0)$ of the integral equation (14) *satisfies the following differential equation*

$$\frac{\partial^2 U(\xi_0,\eta_0)}{\partial\xi_0\partial\eta_0}+\left[-2i\cos\xi\sin\eta+q(\xi-\eta)\right]U(\xi_0,\eta_0)=0 \tag{19}$$

and

$$U(\xi_0,0)=\frac{1}{2}\int_{\xi_0}^{+\infty}q(\xi)\,d\xi. \tag{20}$$

Proof. Differentiating equation (14) and taking into account (13), (18), we find that the function $U(\xi_0,\eta_0)$ is twice differentiable in the domain $\xi_0\ge\eta_0$ and true are the relations

$$\int_{\xi_0}^{+\infty}\left|\xi\frac{\partial U(\xi,\eta)}{\partial\xi}\right|d\xi<\infty,\ \int_{\xi_0}^{+\infty}\left|\xi\frac{\partial U(\xi,\eta)}{\partial\eta}\right|d\xi<\infty,$$

$$\int_{\xi_0}^{+\infty}\left|\frac{\partial^2 U(\xi,\eta)}{\partial\xi^2}\right|d\xi<\infty,\ \int_{\xi_0}^{+\infty}\left|\frac{\partial^2 U(\xi,\eta)}{\partial\eta^2}\right|d\xi<\infty,$$

$$\int_{\xi_0}^{+\infty}\xi^2\left|\frac{\partial^2 U(\xi,\eta)}{\partial\xi\partial\eta}-2i\cos\xi\sin\eta\right|d\xi<\infty. \tag{21}$$

Differentiating equation (14) directly, we have

$$\frac{\partial^2 U(\xi_0,\eta_0)}{\partial\xi_0\partial\eta_0}-2i\cos\xi_0\sin\eta_0 U(\xi_0,\eta_0)=$$

$$=\frac{1}{2}\int_{\xi_0}^{+\infty}\left[\frac{\partial^2 R(\xi_0,\eta_0,\xi,0)}{\partial\xi_0\partial\eta_0}-2i\cos\xi_0\sin\eta_0 R(\xi_0,\eta_0,\xi,0)\right]q(\xi)\,d\xi-$$

$$-\int_{\xi_0}^{+\infty}d\xi\int_0^{\eta_0}\left[\left[\frac{\partial^2 R(\xi_0,\eta_0,\xi,\eta)}{\partial\xi_0\partial\eta_0}-2i\cos\xi_0\sin\eta_0 R(\xi_0,\eta_0,\xi,\eta)\right]\right]U(\xi,\eta)\,q(\xi-\eta)\,d\eta-$$

$$-q(\xi_0-\eta_0)\,U(\xi_0,\eta_0)=-q(\xi_0-\eta_0)\,U(\xi_0,\eta_0)\,.$$

Putting $\eta_0=0$ in (14), we get the result (20).

We now let $\xi_0=\frac{t+x}{2},\eta_0=\frac{t-x}{2}$ and express the function $K^+(x,t)=U(\xi_0,\eta_0)$ as a function of x,t. Then the function $K^+(x,t)$ is twice continuously differentiable. In addition, due to (21), we find that if x is fixed we have the relations

$$\int_{x}^{+\infty}\left|t\frac{\partial K^+(x,t)}{\partial x}\right|dx<\infty,\ \int_{x}^{+\infty}\left|t\frac{\partial K^+(x,t)}{\partial t}\right|dt<\infty,$$

$$\int_x^{+\infty}\left|\frac{\partial^2 K^+(x,t)}{\partial x^2}\right|dt<\infty,\ \int_x^{+\infty}\left|\frac{\partial^2 K^+(x,t)}{\partial t^2}\right|dt<\infty. \tag{22}$$

Moreover, from the two preceding lemmas we get the following lemma. □

Lemma 4. *Suppose $q(x)$ is a continuously differentiable function and satisfies the condition (18). Then the function $K^+(x,t)=U\left(\frac{t+x}{2},\frac{t-x}{2}\right)$ satisfies both the differential equation*

$$\frac{\partial^2 K^+(x,t)}{\partial x^2}-[i\sin x+q(x)]\,K^+(x,t)=\frac{\partial^2 K^+(x,t)}{\partial t^2}-i\sin tK^+(x,t) \tag{23}$$

and the condition

$$K^+(x,x)=\frac{1}{2}\int_x^{+\infty}q(t)\,dt. \tag{24}$$

Furthermore, the following estimate holds

$$\left|K^+(x,t)\right|\le C\sigma_0^+\left(\frac{t+x}{2}\right)e^{C\sigma_1^+(x)}. \tag{25}$$

Now the theorem can be proved. By differentiation from (5), we have

$$f'_+(x,\lambda)=\Psi'_+(x,\lambda)-K^+(x,x)\,\Psi_+(x,\lambda)+\int_x^{+\infty}\frac{\partial K^+(x,t)}{\partial x}\Psi_+(t,\lambda)\,dt, \tag{26}$$

$$f''_+(x,\lambda)=\Psi''_+(x,\lambda)-\frac{dK^+(x,x)}{dx}\Psi_+(x,\lambda)-K^+(x,x)\,\Psi'_+(x,\lambda)-$$

$$-\frac{\partial K^+(x,x)}{\partial x}\Psi_+(x,\lambda)+\int_x^{+\infty}\frac{\partial^2 K^+(x,t)}{\partial x^2}\Psi_+(t,\lambda)\,dt. \tag{27}$$

From

$$-\Psi''_+(x,\lambda)+i\sin x\Psi_+(x,\lambda)=\lambda\Psi_+(x,\lambda) \tag{28}$$

and (5), we have

$$\lambda f_+(x,\lambda)=\lambda\Psi_+(x,\lambda)+\int_x^{+\infty}K^+(x,t)\,i\sin t\Psi_+(t,\lambda)\,dt-$$

$$-\int_x^{+\infty}K^+(x,t)\,\Psi''_+(t,\lambda)\,dt. \tag{29}$$

Hence, integrating by parts, we obtain

$$\int_x^{+\infty}K^+(x,t)\,\Psi''_+(t,\lambda)\,dt=-K^+(x,x)\,\Psi'_+(x,\lambda)-\int_x^{+\infty}\frac{\partial K^+(x,t)}{\partial t}\Psi'_+(t,\lambda)\,dt=$$

$$=-K^+(x,x)\,\Psi'_+(x,\lambda)+\frac{\partial K^+(x,x)}{\partial t}\Psi'_+(x,\lambda)+\int_x^{+\infty}\frac{\partial^2 K^+(x,t)}{\partial t^2}\Psi_+(t,\lambda)\,dt. \tag{30}$$

By virtue of (5) and (27)-(30), we have

$$-f''_+(x,\lambda)+i\sin xf_+(x,\lambda)+q(x)\,f_+(x,\lambda)-\lambda f_+(x,\lambda)=$$

$$=\int_x^{+\infty}\left[\frac{\partial^2 K^+(x,t)}{\partial t^2}-\frac{\partial^2 K^+(x,t)}{\partial x^2}+K^+(x,t)\,(i\sin x+q(x)-i\sin t)\right]\Psi_+(t,\lambda)\,dt+$$

$$+\left[2\frac{dK^+(x,x)}{dx}+q(x)\right]\Psi_+(t,\lambda)+\left[-\Psi''_+(x,\lambda)+i\sin x\Psi_+(x,\lambda)-\lambda\Psi_+(x,\lambda)\right].$$

From the lemma 4 and the last relation, $f_+(x,\lambda)$ satisfies equation (3). In conclusion, we note that the requirement for function $q(x)$ to be differentiable can be omitted, since a function from the class (3) can be approximated by functions from the class (14) and then extrapolation to the limit carried out(see Marchenko (1986)). Thus, the proof of the theorem is complete.

3 Conclusion

The perturbation of a certain Hill equation with a complex periodic potential is considered. Using transformation operators, representations of special solutions to this equation are found. The connection between the perturbation potential and the kernels of transformation operators is indicated. The obtained results obtained can be used to study the inverse scattering problem for the perturbed Hill equation.

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